Recitation 14

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Problems

Problem 1. Let $v_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ be two vectors in $\mathbb{R}^4$. Find two vectors v_3, v_4 such that $\{v_1, v_2, v_3, v_4\}$ is an orthogonal basis of $\mathbb{R}^4$.

Solution. Firt of all notive that $v_1 \cdot v_2 = 0$, so v_1 and v_2 are orthogonal. Let's find all the vectors orthogonal to v_1, v_2. These are all the vectors $x = [x_1,\ x_2,\ x_3,\ x_4]^T$ such that $v_1 \cdot x = v_2 \cdot x = 0$. But these are linear equations, giving us the system

$$\begin{cases} x_1 - 2x_2 + x_3 & = 0 \\ x_1 + x_2 + x_3 + x_4 & = 0 \end{cases}$$

The matrix of this homogeneous system is

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Solving it (row reduction, etc.) gives that x_3, x_4 can be taken to be free variables, and $x_1 = -x_3 - \frac{2}{3}x_4$ and $x_2 = \frac{1}{3}x_4$. Thus a basis for the space of solutions is

$$u_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and } u_2 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 3 \end{bmatrix}$$

Here we re-scaled one of the vectors to get rid of the fractions appearing. By the construction, u_1 and u_2 are orthogonal to the vectors v_1, v_2, but we are not done: u_1 and u_2 are **not orthogonal**, and so can't be taken to be v_3, v_4 we are looking for. So we need to do one more step: use Gramm-Schmidt (or whatever you want) to orthogonalize u_1, u_2. Thus we put $v_3 = u_1$ and

$$v_4 = u_2 - \frac{u_2 \cdot u_1}{u_1 \cdot u_1} u_1 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 3 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix}$$

Very good. Now we have our vectors $v_1 \ldots, v_4$:

$$v_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix}$$

Problem 2. Find an SVD of the matrix

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

Solution. We just do what the doctor told us. First, $A^TA = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$. It's eigenvalues are $\lambda_1 = 9$ and $\lambda_2 = 4$ (note the ordering!). So the singular values are $\sigma_1 = 3$ and $\sigma_2 = 2$. Now we find the corresponding eigenvectors of A^TA. They are $v_1 = e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $v_2 = e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (note the ordering and their length!). So we know the matrix V for the SVD. Namely, $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. To find vectors u_1 and u_2, we compute $u_1 = \frac{1}{\sigma_1} A v_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $u_2 = \frac{1}{\sigma_2} A v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus $U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and we are done with the SVD:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^T$$

Problem 3. Find an SVD of the matrix

$$A = \begin{bmatrix} 3 & -3 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Solution. OK, we really just do the same thing, just following the algorithm. First, $A^TA = \begin{bmatrix} 10 & -8 \\ -8 & 10 \end{bmatrix}$. The characteristic equation is $\lambda^2 - 20\lambda + 36 = 0$, and so $\lambda_1 = 18$ and $\lambda_2 = 2$ (notice the order!). So the singular values are $\sigma_1 = \sqrt{18}$ and $\sigma_2 = \sqrt{2}$. The eigenvectors are $v_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ respectively. Notice that here we normalized our vectors to make them have length 1. Thus we know the matrix $V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$. To find U, we get $u_1 = \frac{1}{\sqrt{18}} A v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $u_2 = \frac{1}{\sqrt{2}} A v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Since U should be a 3×3 orthogonal matrix, and we only have two columns u_1, u_2, we need to find one more vector u_3 of length 1 which would be orthogonal to u_1, u_2. For that, you can either do the same thing as in Problem 1, or just notice that $u_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ will do the job (it's easy to see here, since u_1, u_2 are just two standard basis vectors). Thus, we are done with finding U, and hence with the SVD:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -3 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T$$

Problem 4. Find the pseudoinverse of the matrix A from Problem 3.

Solution. Since you didn't cover this stuff on the lectures, it doesn't matter.

Problem 5. Which of the following quadratic forms are positive-definite, negative-definite, or neither? Which are semi-definite?

1. $Q_1(x) = 3x_1^2 - 2x_1x_2 + x_2^2$ on $\mathbb{R}^2$;
2. $Q_2(x) = 3x_1^2 - 2x_1x_2 + x_2^2$ on $\mathbb{R}^3$;
3. $Q_3(x) = 6x_1x_2 + 4x_1x_3$ on $\mathbb{R}^3$;
4. $Q_4(x) = -x_1^2 + 2x_1x_2$ on $\mathbb{R}^2$.

Solution. The secret is: you find the matrix of the quadratic form, find its eigenvalues and look at their signs. If all eigenvalues are positive (resp. negative) then the form is positive (resp. negative) definite. If all eigenvalues are non-strictly positive (negative) then the form is positive (negative) semi-definite.
For $Q_1(x) = 3x_1^2 - 2x_1x_2 + x_2^2$ on $\mathbb{R}^2$, the matrix is

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$$

The characteristic equation is $\lambda^2 - 4\lambda + 2 = 0$, and so the eigenvalues are $2 \pm \sqrt{2}$, both strictly positive. Thus $Q_1(x) = 3x_1^2 - 2x_1x_2 + x_2^2$ on $\mathbb{R}^2$ is positive definite.

If we consider $Q_1(x) = 3x_1^2 - 2x_1x_2 + x_2^2$ on $\mathbb{R}^3$, the matrix is

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the eigenvalues are $0, 2+\sqrt{2}, 2-\sqrt{2}$, non-strictly positive. So Q is a positive semi-definite matrix.

The matrix of the form $Q_3(x) = 6x_1x_2 + 4x_1x_3$ on $\mathbb{R}^3$ is

$$A = \begin{bmatrix} 0 & 3 & 2 \\ 3 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

and the characteristic equation is $\lambda^3 - 13\lambda = 0$. Hence the eigenvalues are $0, \sqrt{13}, -\sqrt{13}$. So this form is neither positive nor negative (semi-)definite.

For $Q_4(x) = -x_1^2 + 2x_1x_2$ on $\mathbb{R}^2$, the matrix is

$$A = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

whose eigenvalues are $\frac{1\pm\sqrt{5}}{2}$, and so Q is again neither positive nor negative definite.

Problem 6. Let $A = U\Sigma V^T$ be a SVD of an $m \times n$ matrix A.

Prove that if A is a square matrix, then $|\det A|$ is the product of singular values of A.

Prove that the columns of V are eigenvectors of A^TA, and the columns of U are eigenvectors of AA^T.

Solution. For any **orthogonal** matrix U, $\det U = \pm 1$. Thus $\det A = \det U \cdot \det \Sigma \cdot \det V^T = \pm \det \Sigma$. Taking absolute values gives $|\det A| = |\det \Sigma| = \det \Sigma$, where the last equality holds since all the singular values by definition are non-negative.
If $A = U\Sigma V^T$, then $A^TAV = V\Sigma^T\Sigma V^TV = V\Sigma^T\Sigma$ because $V^TV = I$ by the definition of orthogonal matrix V. Note also that $\Sigma^T\Sigma$ as a **diagonal square** matrix. Looking at i-th column, we get $A^TAv_i = \sigma_i^2 v_i$. That's exactly what we wanted to get.
Similarly for AA^T and columns of U.

Problem 7. Prove that for any $m \times n$ matrix A defining $A\colon \mathbb{R}^n \to \mathbb{R}^m$, you can always find a basis of $\mathbb{R}^n$ and a basis of $\mathbb{R}^m$, relative to which the matrix A becomes

$$\Sigma' = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

where I is $r \times r$ **identity** matrix. In other words, you can always find $n \times n$ matrix P and $m \times m$ matrix Q such that $A = Q\Sigma' P^{-1}$.

Solution. Let's pick $v_1, \dots, v_r \in \mathbb{R}^m$ to be a basis of $Col(A)$. We can complete it to a basis $v_1, \dots, v_r, v_{r+1}, \dots, v_m$ of the whole $\mathbb{R}^m$. By definition of $Col(A)$, there are some vectors $u_1, \dots, u_r \in \mathbb{R}^n$ that are mapped to $v_1, \dots, v_r$, i.e. $v_1 = Au_1, \dots, v_r = Au_r$. They are linearly independent, since $Au_1, \dots, Au_r$ are. Let $u_{r+1}, \dots, u_n$ to be a basis of $Nul(A)$. Note that $u_1, \dots, u_r, u_{r+1}, \dots, u_n$ is a basis of $\mathbb{R}^n$ (think why!).
In this basis, A takes the required form. Check that this is indeed the case! You need to take u_i's, see where they go under A, and find the weights of Au_i in terms of $v_1, \dots, v_n$.